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# The Casimir effect in a deformed field

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Received 23 December 2007, in final form 24 March 2008

Published 18 April 2008

Online at [stacks.iop.org/JPhysA/41/185402](http://stacks.iop.org/JPhysA/41/185402)

## Abstract

The Casimir energy is calculated in one-, two- and three-dimensional spaces for the field with generalized coordinates and momenta satisfying the deformed Poisson brackets leading to the minimal length.

PACS numbers: 03.70.+k, 12.20.-m

## 1. Initial equations

Starting from the seminal work [1], the Casimir effect has been studied in the course of several decades for various physical systems, ranging from problems in solid-state physics to those in cosmology. Some interest has recently arisen in the study of the Casimir effect in the so-called deformed spaces, i.e., in the spaces with deformed Poisson brackets between spatial coordinates and momentum, in particular, in those leading to the minimal lengths [2–4]. In the paper [2] the effect of minimal length in the Casimir–Polder interaction between neutral atoms in the first-order correction term in the minimal uncertainty parameter was studied. The influence of the minimal length on the Casimir energy between two plates was studied in [3, 4]. Note also the papers [5, 6] where the Casimir effect was studied in the case of a  $\kappa$ -deformed Poincaré algebra.

In our work, we study the Casimir effect for the deformed electro-magnetic field with the Hamiltonian

$$\hat{H} = \sum_{\mathbf{k}} \sum_{\alpha} \left( \frac{\hat{P}_{\mathbf{k},\alpha}^2}{2} + \frac{\omega_k^2 \hat{Q}_{\mathbf{k},\alpha}^2}{2} \right), \quad (1)$$

where  $\mathbf{k}$  is the wave vector,  $\alpha$  is the polarization index, the frequency  $\omega_k = ck$ , where  $c$  is the speed of light in vacuum, and  $\hat{Q}_{\mathbf{k},\alpha}$ ,  $\hat{P}_{\mathbf{k},\alpha}$  are operators of generalized coordinates and momenta that satisfy deformed Poisson brackets:

$$\hat{Q}_{\mathbf{k},\alpha} \hat{P}_{\mathbf{k},\alpha} - \hat{P}_{\mathbf{k},\alpha} \hat{Q}_{\mathbf{k},\alpha} = i\hbar(1 + \beta \hat{P}_{\mathbf{k},\alpha}^2), \quad (2)$$

where  $\beta \geq 0$  is the deformation parameter, and all other commutators are equal to zero. In our case the configuration space is undeformed. We note that the deformation parameter can also depend on  $\mathbf{k}$  and  $\alpha$ . In this work, however, we consider the case  $\beta = \text{const}$ .

As is known, such commutation relations lead to the existence of the minimal length  $\sqrt{\langle \hat{Q}_{\mathbf{k},\alpha}^2 \rangle} = \hbar\sqrt{\beta}$  in the space of field coordinates [7]. Obviously, in this case one has the deformation of the field itself rather than the space deformation. Because for deformed fields  $\langle \hat{Q}_{\mathbf{k},\alpha}^2 \rangle \geq \hbar^2\beta$  one can say that the fluctuations are stronger in the case of deformed fields. Note that here we have deformation of fields.

Proceeding now to new operators  $\hat{q}_{\mathbf{k},\alpha}, \hat{p}_{\mathbf{k},\alpha}$ ,

$$\hat{q}_{\mathbf{k},\alpha} = \hat{Q}_{\mathbf{k},\alpha}, \quad \hat{p}_{\mathbf{k},\alpha} = \frac{1}{\sqrt{\beta}} \tan(\hat{p}_{\mathbf{k},\alpha}\sqrt{\beta}), \quad (3)$$

it is easy to show that they are canonically conjugated, i.e.,

$$\hat{q}_{\mathbf{k},\alpha} \hat{p}_{\mathbf{k},\alpha} - \hat{p}_{\mathbf{k},\alpha} \hat{q}_{\mathbf{k},\alpha} = i\hbar, \quad (4)$$

and, hence, the Hamiltonian reads

$$\hat{H} = \sum_{\mathbf{k}} \sum_{\alpha} \left( \frac{\omega_{\mathbf{k}}^2 \hat{q}_{\mathbf{k},\alpha}^2}{2} + \frac{\tan^2(\hat{p}_{\mathbf{k},\alpha}\sqrt{\beta})}{2\beta} \right). \quad (5)$$

The equations of motion of the deformed field have been studied in [8], where the Hamiltonian was presented in the form of the expansion in powers of the ordinary creation and annihilation operators. Generally speaking, the field equations are nonlinear and are usually analyzed by means of the perturbation theory. We note that the nonlinear field described by the  $q$ -oscillators which also leads to minimal length in the space of field has been studied in [9]. The model given by equation (5) is strongly nonlinear for the sufficiently large deformation parameter  $\beta$ . Nevertheless, it is simple enough to be treated analytical or to get numerical results.

The energy levels of the harmonic oscillator (5) with the commutation relation (4), or Hamiltonian (1) with the commutation relation (2), are well known [7, 10]. One therefore easily finds for the energy levels of the deformed field:

$$E_{\dots, N_{\mathbf{k},\alpha}, \dots} = \sum_{\mathbf{k}} \sum_{\alpha} \hbar\omega_{\mathbf{k}} \left[ \left( N_{\mathbf{k},\alpha} + \frac{1}{2} \right) \sqrt{1 + \left( \frac{\beta\hbar\omega_{\mathbf{k}}}{2} \right)^2} + \frac{\beta\hbar\omega_{\mathbf{k}}}{2} \left( N_{\mathbf{k},\alpha}^2 + N_{\mathbf{k},\alpha} + \frac{1}{2} \right) \right], \quad (6)$$

where  $N_{\mathbf{k},\alpha} = 0, 1, 2, \dots$  are quantum numbers.

The energy of the vacuum state ( $N_{\mathbf{k},\alpha} = 0$ ) is

$$E_{\dots, 0, \dots} = \sum_{\mathbf{k}} \sum_{\alpha} \frac{\hbar\omega_{\mathbf{k}}}{2} \left( \sqrt{1 + \left( \frac{\beta\hbar\omega_{\mathbf{k}}}{2} \right)^2} + \frac{\beta\hbar\omega_{\mathbf{k}}}{2} \right). \quad (7)$$

The aim of the present work is to calculate the Casimir energy for the deformed space as a function of the deformation parameter  $\beta$ ; to accomplish this task, we shall start from equation (7). We consider Casimir effect in one-, two- and three-dimensional cases separately because each of these cases has some peculiarity related to different boundary conditions for different dimensions.

## 2. Casimir energy in the one-dimensional case

Let us first consider the one-dimensional (1D) case. The scalar field is located on the segment of length  $a$  along the  $x$ -axis between the points  $x = 0$  and  $x = a$ , at which the field is zero. For such conditions, the wave vector in equation (7)  $k = \pi n/a, n = 1, 2, \dots$ , and there is

no polarization. By definition, the Casimir energy  $\varepsilon$  is the difference of the vacuum energy density (7) in volume  $a$  and infinite volume  $L$ :

$$\varepsilon = \frac{1}{a^2} \sum_{n=1}^{\infty} \frac{\hbar c \pi n}{2} \left( \sqrt{1 + \left(\frac{\beta \hbar c}{2a} n\right)^2} + \frac{\beta \hbar c}{2a} n \right) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hbar c |k|}{2} \left( \sqrt{1 + \left(\frac{\beta \hbar c}{2} k\right)^2} + \frac{\beta \hbar c}{2} |k| \right) dk. \tag{8}$$

To regularize this expression one can introduce a cut-off function  $e^{-\nu k}$ ,  $\nu > 0$ , which ensures both the convergence of the summation over  $n$  and the integration over  $k$ . Further, we make the change of variable  $k = \pi n/a$  to get

$$\varepsilon \underset{\nu \rightarrow 0}{=} \frac{\hbar c \pi}{2a^2} \left[ \sum_{n=1}^{\infty} e^{-\nu n} n \left( \sqrt{1 + (\beta^* n)^2} + \beta^* n \right) - \int_0^{\infty} e^{-\nu n} n \left( \sqrt{1 + (\beta^* n)^2} + \beta^* n \right) dn \right], \tag{9}$$

where the dimensionless deformation parameter is

$$\beta^* = \beta \frac{\hbar c \pi}{2a}.$$

We then apply the Abel–Plana formula [11] to calculate the sum over  $n$  in equation (9):

$$\sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(n) dn - \frac{f(0)}{2} + i \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt. \tag{10}$$

Equation (10) can be applied for functions with  $f(0) = 0$  as well as for function with  $f(0) \neq 0$ . Note that we can rewrite this equation in the following way. The sum on the left-hand side of equation (10) can be written as  $\sum_{n=0}^{\infty}$  and then the right-hand side of this equation will contain  $f(0)/2$  instead of  $-f(0)/2$ . In our case

$$f(n) = e^{-\nu n} n \left( \sqrt{1 + (\beta^* n)^2} + \beta^* n \right). \tag{11}$$

Using equations (10) and (11), and performing the integration over  $t$  (note that the branching of the integrand at  $t = 1/\beta^*$  must be taken into account) and  $\nu \rightarrow 0$  one gets

$$\varepsilon = -\frac{\hbar c \pi}{a^2} \int_0^{1/\beta^*} \frac{t \sqrt{1 - (\beta^* t)^2}}{e^{2\pi t} - 1} dt. \tag{12}$$

Note that for  $\beta^* = 0$  we recover the well-known result for the Casimir energy in the one-dimensional case for the undeformed field [10, 12]:

$$\varepsilon = -\frac{\hbar c \pi}{24a^2}. \tag{13}$$

Let us now find the expansion of  $\varepsilon$  for  $\beta^* > 1$ . For this purpose we rewrite equation (12) as follows:

$$\varepsilon = -\frac{\hbar c}{2\beta^* a^2} \int_0^1 \sqrt{1 - x^2} \frac{2\pi x/\beta^*}{e^{2\pi x/\beta^*} - 1} dx. \tag{14}$$

We then expand the second factor under the integral in a power series in  $2\pi x/\beta^*$  and use the definition of the Bernoulli numbers  $B_n$  [13, 14] to obtain

$$\varepsilon = -\frac{\hbar c}{2\beta^* a^2} \sum_{n=0}^{\infty} \frac{B_n}{n!} \left(\frac{2\pi}{\beta^*}\right)^n \int_0^1 x^n \sqrt{1 - x^2} dx. \tag{15}$$

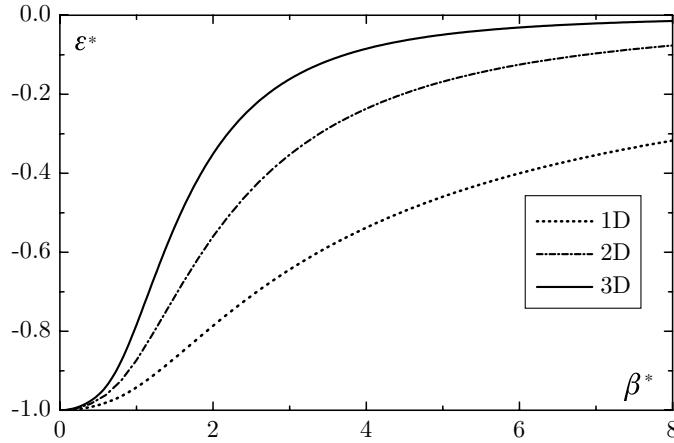


Figure 1. Casimir energy as a function of the deformation parameter.

The integral in equation (15) is the Euler B-function, and the Bernoulli numbers are expressed via Riemann’s  $\zeta$ -function [13, 14]. As a result we obtain

$$\begin{aligned} \varepsilon &= \frac{\hbar c \pi}{2a^2} \left( \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1/2)\zeta(2n)}{(n + 1)!2\sqrt{\pi}\beta^{*2n+1}} - \frac{1}{4\beta^*} + \frac{1}{3\beta^{*2}} \right) \\ &= -\frac{\hbar c \pi}{24a^2} \left( \frac{3}{\beta^*} - \frac{4}{\beta^{*2}} + \frac{\pi^2}{4\beta^{*3}} - \frac{\pi^4}{120\beta^{*5}} + \frac{\pi^6}{2016\beta^{*7}} + \dots \right). \end{aligned} \tag{16}$$

It is interesting to note that one can come to the same result directly from equation (9). For this purpose, we extract the asymptotic term  $2\beta^*n^2 + 1/2\beta^*$  for  $n \rightarrow \infty$  in the expression in parentheses in equation (9) and write it as follows:

$$\begin{aligned} \varepsilon \underset{\nu \rightarrow 0}{=} \frac{\hbar c \pi}{2a^2} \left\{ \sum_{n=1}^{\infty} e^{-\nu n} \left[ \beta^* n^2 \left( \sqrt{1 + \frac{1}{(\beta^* n)^2}} - 1 \right) - \frac{1}{2\beta^*} \right] \right. \\ \left. - \int_0^{\infty} e^{-\nu n} \left[ \beta^* n^2 \left( \sqrt{1 + \frac{1}{(\beta^* n)^2}} - 1 \right) - \frac{1}{2\beta^*} \right] dn \right. \\ \left. + \sum_{n=1}^{\infty} e^{-\nu n} \left( 2\beta^* n^2 + \frac{1}{2\beta^*} \right) - \int_0^{\infty} e^{-\nu n} \left( 2\beta^* n^2 + \frac{1}{2\beta^*} \right) dn \right\}. \end{aligned} \tag{17}$$

The first integral in equation (17) yields  $(-1/3\beta^{*2})$  for  $\nu \rightarrow 0$ , while the difference between the two last terms gives  $(-1/4\beta^*)$ . We therefore have

$$\varepsilon = \frac{\hbar c \pi}{2a^2} \left\{ \sum_{n=1}^{\infty} \left[ \beta^* n^2 \left( \sqrt{1 + \frac{1}{(\beta^* n)^2}} - 1 \right) - \frac{1}{2\beta^*} \right] - \frac{1}{4\beta^*} + \frac{1}{3\beta^{*2}} \right\}. \tag{18}$$

Because the sum over  $n$  converges, the cut-off parameter  $\nu$  can be set to zero. By expanding (formally) the square root in a power series in  $1/\beta^{*2}$  and using the definition of the Riemann  $\zeta$ -function, we come to equation (16).

In order to make computer calculations more convenient, formula (18) can be transformed to the following form:

$$\varepsilon = -\frac{\hbar c \pi}{2\beta^* a^2} \left[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(\sqrt{1 + (\beta^* n)^2} + \beta^* n)^2} + \frac{1}{4} - \frac{1}{3\beta^*} \right]. \tag{19}$$

We shall note, however, that the calculation of  $\varepsilon$  using equation (12) is more effective in comparison with the summation in equation (19). This is because of the fast convergence of the integral in equation (12).

Let us now make the expansion of the Casimir energy for small values of the deformation parameter. One can apply the Euler–Maclaurin formula directly to (9), as Casimir did in [1]. However, the result can be obtained in a more convenient way if we rewrite equation (9) as follows:

$$\begin{aligned} \varepsilon &\underset{\nu \rightarrow 0}{=} -\frac{\hbar c \pi}{2a^2} \frac{d}{d\nu} \sqrt{1 + \left(\beta^* \frac{d}{d\nu}\right)^2} \left( \sum_{n=1}^{\infty} e^{-\nu n} - \int_0^{\infty} e^{-\nu n} dn \right) \\ &\underset{\nu \rightarrow 0}{=} -\frac{\hbar c \pi}{2a^2} \frac{d}{d\nu} \sqrt{1 + \left(\beta^* \frac{d}{d\nu}\right)^2} \left( \frac{1}{e^\nu - 1} - \frac{1}{\nu} \right) \\ &\underset{\nu \rightarrow 0}{=} -\frac{\hbar c \pi}{2a^2} \frac{d}{d\nu} \sqrt{1 + \left(\beta^* \frac{d}{d\nu}\right)^2} \sum_{n=1}^{\infty} \frac{B_n}{n!} \nu^{n-1}. \end{aligned} \tag{20}$$

Here again we have used the definition of the Bernoulli numbers  $B_n$  [13, 14]. We now make a formal expansion of the square root in powers of  $\beta^{*2}$  and take the derivatives with respect to  $\nu$ , with subsequently taking the limit  $\nu \rightarrow 0$ , to find, after a few simple transformations, the following asymptotic expansion:

$$\begin{aligned} \varepsilon &= -\frac{\hbar c \pi}{2a^2} \left( \frac{1}{12} + \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n - 1/2)}{(n + 1)! 4\sqrt{\pi}} B_{2n+2} \beta^{*2n} \right) \\ &= -\frac{\hbar c \pi}{24a^2} \left( 1 - \frac{\beta^{*2}}{20} - \frac{\beta^{*4}}{168} - \frac{\beta^{*6}}{320} - \frac{5}{1408} \beta^{*8} - \dots \right). \end{aligned} \tag{21}$$

The result of the calculations of the Casimir energy, equations (12) and (19),

$$\varepsilon_{1D}^* = \varepsilon \left/ \frac{\hbar c \pi}{24a^2} \right.$$

is presented in figure 1 (curve 1D). As one can see, the Casimir energy for all values of the parameter  $\beta^*$  is larger than for the undeformed field and goes to zero for  $\beta^* \rightarrow \infty$ . Therefore, the deformation leads to the decrease of the attraction of the domain boundaries localizing the field caused by the polarization of vacuum.

### 3. The three-dimensional case

We now calculate the Casimir energy in the three-dimensional (3D) case. The field is located between two infinite parallel plates separated by a distance  $a$  and is set to zero at the plates. The wave vector  $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ ,  $-\infty < k_x < \infty$ ,  $-\infty < k_y < \infty$ ,  $k_z = \pi n/a$ ,  $n = 0, 1, 2, \dots$ , and the sum

$$\frac{1}{L^2} \sum_{\mathbf{k}} \dots \xrightarrow{L \rightarrow \infty} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \sum_{n=0}^{\infty} \dots$$

Unlike as in the 1D case, in the 3D case the term  $n = 0$  must be taken into account in the expression for the ground-state energy (7). However, the polarization index  $\alpha$  for  $n = 0$  and  $D = 3$  has only one value because of the field transversality. From the definition of the Casimir energy, one has

$$\begin{aligned} \varepsilon = & \frac{1}{a} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \frac{\hbar c}{2} \sqrt{k_x^2 + k_y^2} \left[ \sqrt{1 + \left(\frac{\beta\hbar c}{2}\right)^2 (k_x^2 + k_y^2)} + \frac{\beta\hbar c}{2} \sqrt{k_x^2 + k_y^2} \right] \\ & + \frac{1}{a} \sum_{\alpha} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \sum_{n=1}^{\infty} \frac{\hbar c}{2} \sqrt{k_x^2 + k_y^2 + \left(\frac{\pi n}{a}\right)^2} \\ & \times \left[ \sqrt{1 + \left(\frac{\beta\hbar c}{2}\right)^2 \left(k_x^2 + k_y^2 + \left(\frac{\pi n}{a}\right)^2\right)} + \frac{\beta\hbar c}{2} \sqrt{k_x^2 + k_y^2 + \left(\frac{\pi n}{a}\right)^2} \right] \\ & - \sum_{\alpha} \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \frac{\hbar c}{2} \sqrt{k_x^2 + k_y^2 + k_z^2} \\ & \times \left[ \sqrt{1 + \left(\frac{\beta\hbar c}{2}\right)^2 (k_x^2 + k_y^2 + k_z^2)} + \frac{\beta\hbar c}{2} \sqrt{k_x^2 + k_y^2 + k_z^2} \right]. \end{aligned} \quad (22)$$

The integration is performed in polar coordinates,  $q = \sqrt{k_x^2 + k_y^2}$ . We then change the variables  $k = \sqrt{q^2 + (\pi n/a)^2}$  (replacing  $k$  with  $q$ ) and introduce a cut-off function  $e^{-\nu q}$  to get the following result:

$$\begin{aligned} \varepsilon \underset{\nu \rightarrow 0}{=} & \frac{\hbar c}{2\pi a} \left\{ \frac{1}{2} \int_0^{\infty} dq q^2 e^{-\nu q} \left[ \sqrt{1 + \left(\frac{\beta\hbar c q}{2}\right)^2} + \frac{\beta\hbar c q}{2} \right] \right. \\ & \left. + \sum_{n=1}^{\infty} \int_{\pi n/2}^{\infty} dq q^2 e^{-\nu q} \left[ \sqrt{1 + \left(\frac{\beta\hbar c q}{2}\right)^2} + \frac{\beta\hbar c q}{2} \right] \right\} \\ & - \frac{\hbar c}{2\pi^2} \int_0^{\infty} dk_z \int_{k_z}^{\infty} dq q^2 e^{-\nu q} \left[ \sqrt{1 + \left(\frac{\beta\hbar c q}{2}\right)^2} + \frac{\beta\hbar c q}{2} \right]. \end{aligned} \quad (23)$$

The Abel–Plana formula (10) with the function

$$f(n) = \int_{\pi n/2}^{\infty} q^2 e^{-\nu q} \left[ \sqrt{1 + \left(\frac{\beta\hbar c q}{2}\right)^2} + \frac{\beta\hbar c q}{2} \right] dq \quad (24)$$

is then applied to the second term in brackets. Noting that the term  $f(0)$  in equation (10) cancels the first term in equation (23), and the last term in (23) cancels the first integral from the Abel–Plana formula (10), we obtain, after simple transformations

$$\varepsilon = -\frac{\hbar c \pi^2}{2a^4} \int_0^{1/\beta^*} dt \frac{1}{e^{2\pi t} - 1} \int_{-t}^t x^2 \sqrt{1 - (\beta^* x)^2} dx. \quad (25)$$

Finally, taking the integral over  $x$  we find

$$\varepsilon = -\frac{\hbar c \pi^2}{8\beta^{*2} a^4} \int_0^{1/\beta^*} \frac{(2\beta^{*2} t^3 - t) \sqrt{1 - (\beta^* t)^2} + \arcsin(\beta^* t) / \beta^*}{e^{2\pi t} - 1} dt. \quad (26)$$

For  $\beta^* \rightarrow 0$  the numerator of the integrand in (26) goes to  $8\beta^{*2}t^3/3$  and, hence,

$$\varepsilon = -\frac{\hbar c \pi^2}{3a^4} \int_0^\infty \frac{t^3}{e^{2\pi t} - 1} dt = -\frac{\hbar c \pi^2}{720a^4}. \tag{27}$$

This is the Casimir result [1].

We shall now find the expansion of the Casimir energy in powers of  $1/\beta^*$  (for  $\beta^* \neq 0$ ). Expression (26) after the change of variable  $x = \beta^*t$  is

$$\varepsilon = -\frac{\hbar c \pi}{16a^4 \beta^{*3}} \int_0^1 \left[ (2x^2 - 1)\sqrt{1 - x^2} + \frac{\arcsin x}{x} \right] \frac{2\pi x/\beta^*}{e^{2\pi x/\beta^*} - 1} dx. \tag{28}$$

The expansion of the second factor under the integral in powers of  $2\pi x/\beta^*$  [13, 14] leads to

$$\varepsilon = -\frac{\hbar c \pi}{16a^4 \beta^{*3}} \sum_{n=0}^\infty \frac{B_n}{n!} \left(\frac{2\pi}{\beta^*}\right)^n I(n), \tag{29}$$

where

$$I(n) = \int_0^1 x^n \left[ (2x^2 - 1)\sqrt{1 - x^2} + \frac{\arcsin x}{x} \right] dx. \tag{30}$$

The integration in (30) gives

$$\begin{aligned} I(0) &= \frac{\pi}{2} \left( \ln 2 - \frac{1}{4} \right), \\ I(1) &= \frac{\pi}{2} - \frac{16}{15}, \\ I(2n) &= \frac{\pi}{4n} - \frac{2\Gamma(3/2)\Gamma(n+3/2)}{n\Gamma(n+3)}. \end{aligned}$$

We need the values of  $I(n)$  for  $n > 1$  only for even  $n$  because  $B_{2n+1} = 0$  for  $n > 1$ . Using the relation between the Bernoulli numbers and the Riemann  $\zeta$ -function, we find for  $\beta^* \neq 0$ , after a few transformations:

$$\begin{aligned} \varepsilon &= -\frac{\hbar c \pi^2}{16a^4 \beta^{*3}} \left\{ \frac{1}{2} \left( \ln 2 - \frac{1}{4} \right) - \frac{1}{\beta^*} \left( \frac{\pi}{2} - \frac{16}{15} \right) \right. \\ &\quad \left. + \sum_{n=1}^\infty (-)^{n-1} \frac{\zeta(2n)}{\beta^{*2n} 2n} \left[ 1 - \frac{(2n+1)!}{2^{2n-1} (n!)^2 (n+1)(n+2)} \right] \right\} \\ &= -\frac{\hbar c \pi^2}{16a^4 \beta^{*3}} \left[ \frac{1}{2} \left( \ln 2 - \frac{1}{4} \right) - \frac{1}{\beta^*} \left( \frac{\pi}{2} - \frac{16}{15} \right) + \frac{\pi^2}{24\beta^{*2}} + \dots \right]. \tag{31} \end{aligned}$$

We now calculate the asymptotic expansion of  $\varepsilon$  for small  $\beta$  proceeding from formula (23), which we rewrite as follows:

$$\begin{aligned} \varepsilon &\underset{\nu \rightarrow 0}{=} \frac{d^2}{d\nu^2} \left[ \sqrt{1 + \left( \frac{\beta \hbar c}{2} \frac{d}{d\nu} \right)^2} - \frac{\beta \hbar c}{2} \frac{d}{d\nu} \right] \\ &\quad \times \frac{1}{\nu} \left( \frac{\hbar c}{4\pi a} + \frac{\hbar c}{2\pi a} \sum_{n=1}^\infty e^{-\nu \pi n/a} - \frac{\hbar c}{2\pi^2} \int_0^\infty dk_z e^{-\nu k_z} \right). \tag{32} \end{aligned}$$

Changing the cut-off parameter  $\nu \pi/a \rightarrow \nu$  we find after simple calculations

$$\varepsilon \underset{\nu \rightarrow 0}{=} \frac{\hbar c \pi^2}{2a^4} \frac{d^2}{d\nu^2} \left[ \sqrt{1 + \left( \beta^* \frac{d}{d\nu} \right)^2} - \beta^* \frac{d}{d\nu} \right] \frac{1}{\nu} \left( \frac{1}{2} + \frac{1}{e^\nu - 1} - \frac{1}{\nu} \right). \tag{33}$$



Expanding the above expression in a power series in  $\beta^*$  and using the procedure from the previous section we finally obtain

$$\varepsilon = \frac{\hbar c \pi^2}{2a^4} \left( -\frac{1}{360} + \sum_{k=1} \frac{\Gamma(k-1/2)(-)^{k-1} B_{2k+4}}{k! \sqrt{\pi} 2(2k+4)(2k+3)} \beta^{*2k} \right). \quad (34)$$

A few first terms read

$$\varepsilon = -\frac{\hbar c \pi^2}{720a^4} \left( 1 - \frac{1}{7} \beta^{*2} - \frac{3}{112} \beta^{*4} - \frac{5}{264} \beta^{*6} + \dots \right). \quad (35)$$

The first term gives the Casimir result [1], and the remaining terms take into account field deformation. As in the one-dimensional case, the deformation leads to the repulsion and thus reduces the attraction between the plates.

The Casimir energy,

$$\varepsilon_{3D}^* = \varepsilon \left/ \frac{\hbar c \pi^2}{720a^4} \right.,$$

is shown in figure 1 (curve 3D); to calculate it numerically, we have used equation (26).

#### 4. The two-dimensional case

On the two-dimensional (2D)  $xy$  plane the field is located in an infinite stripe of width  $a$  between  $y = 0$  and  $y = a$ ; at  $y = 0$  and  $y = a$  the field is zero. In fact in  $D = 2$  we have a scalar field satisfying Dirichlet condition on the boundary. From the field transversality only one polarization remains; it is perpendicular to the wave vector with the components  $k_x, k_y$ :  $-\infty < k_x < \infty, k_y = \pi n/a, n = 1, 2, 3, \dots$ . The value  $n = 0$  is not taken into account because at  $n = 0$  the field is zero owing to its transversality.

The Casimir energy is

$$\begin{aligned} \varepsilon = & \frac{1}{2\pi a} \int_{-\infty}^{\infty} dk_x \sum_{n=1}^{\infty} \frac{\hbar c}{2} \sqrt{k_x^2 + \left(\frac{\pi}{a}n\right)^2} \\ & \times \left[ \sqrt{1 + \left(\frac{\beta \hbar c}{2}\right)^2 \left(k_x^2 + \left(\frac{\pi}{a}n\right)^2\right)} + \frac{\beta \hbar c}{2} \sqrt{k_x^2 + \left(\frac{\pi}{a}n\right)^2} \right] \\ & - \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \frac{\hbar c}{2} \sqrt{k_x^2 + k_y^2} \\ & \times \left[ \sqrt{1 + \left(\frac{\beta \hbar c}{2}\right)^2 (k_x^2 + k_y^2)} + \frac{\beta \hbar c}{2} \sqrt{k_x^2 + k_y^2} \right]. \quad (36) \end{aligned}$$

Because at  $n = 0$  the field is zero, the function  $f(n)$  must be determined in such a way that  $f(0) = 0$ . This determination can be made through the cut-off function  $e^{-\nu/n}$ . The same determination must be made also in the second term of equation (36). This corresponds to the case that the field is defined in a large volume tending to infinity and is zero on the surface that bounds the volume.

Using the Abel–Plana formula after transformations similar to those made in the previous section we obtain from equation (36):

$$\varepsilon = -\frac{\hbar c \pi}{a^3} \int_0^{1/\beta^*} \frac{t^2 dt}{e^{2\pi t} - 1} \int_0^1 x^2 \sqrt{\frac{1 - (\beta^* t)^2 x^2}{1 - x^2}} dx. \quad (37)$$

For  $\beta = 0$  the integral over  $x$  is easily calculated (it is  $\pi/4$ ) and from (37) one obtains the well-known result [10, 12]:

$$\varepsilon = -\frac{\hbar c}{16\pi a^3} \zeta(3).$$

For  $\beta \neq 0$  the integral over  $x$  in (37) cannot be expressed in terms of elementary functions. It can, however, be reduced to the complete elliptic integrals  $E(k)$  and  $K(k)$  [13, 14]:

$$\int_0^1 x^2 \sqrt{\frac{1 - (\beta^* t)^2 x^2}{1 - x^2}} dx = \frac{1}{3} \left(2 - \frac{1}{k^2}\right) E(k) + \frac{1}{3} \left(\frac{1}{k^2} - 1\right) K(k), \quad (38)$$

where the modulus  $k = \beta^* t$ .

Let us now find the expansion of  $\varepsilon$  for small  $\beta$ . For this purpose, one can use the known expansions of the elliptic integrals for small values of the modulus  $k$  or proceed directly from equation (37). We take the second route and expand the square root in the numerator of the integrand in equation (37) in power series in  $\beta^*$ ; one has

$$\varepsilon = -\frac{\hbar c \pi}{a^3} \sum_{m=0}^{\infty} \frac{\Gamma(3/2)(-\beta^{*2})^m}{m! \Gamma(3/2 - m)} \int_0^1 dx \frac{x^{2m+2}}{\sqrt{1-x^2}} \int_0^{1/\beta^*} \frac{t^{2m+2}}{e^{2\pi t} - 1} dt. \quad (39)$$

The integral over  $x$  is  $\Gamma(m+3/2)\Gamma(1/2)/2\Gamma(m+2)$ . The second integral over  $t$  can be split into two parts: from 0 to  $\infty$  and from  $\infty$  to  $1/\beta^*$ . The first one yields  $\Gamma(2m+3)\zeta(2m+3)/2\pi^{2m+3}$  and the second one gives the terms proportional to  $e^{-1/\beta^*}$ , which can be neglected in the limit  $\beta^* \rightarrow 0$ . We therefore obtain

$$\begin{aligned} \varepsilon &= -\frac{\hbar c \pi}{a^3} \left[ \frac{\zeta(3)}{16\pi^2} - \sum_{m=1} \frac{\Gamma(3/2)\Gamma(m-1/2)\Gamma(1/2)\Gamma(m+3/2)}{2\pi \Gamma(m+2)m!} \right. \\ &\quad \left. \times \frac{\Gamma(2m+3)\zeta(2m+3)}{(2\pi)^{2m+3}} \beta^{*2m} + O(e^{-1/\beta^*}) \right] \\ &= -\frac{\hbar c \pi}{a^3} \left[ \frac{\zeta(3)}{16\pi^2} - \frac{9\zeta(5)}{128\pi^4} \beta^{*2} - \frac{225\zeta(7)}{2048\pi^6} \beta^{*4} + \dots \right]. \end{aligned} \quad (40)$$

Let us calculate the expansion of the Casimir energy in inverse powers of the deformation parameter. It is more convenient to proceed directly from formula (37). By changing the variable of integration  $y = \beta^* t$  and expanding the integrand in a power series in  $1/\beta^*$ , as has been done in the previous cases, we get

$$\varepsilon = -\frac{\hbar c}{2a^3 \beta^{*2}} \sum_{n=0}^{\infty} \frac{B_n}{n!} \left(\frac{2\pi}{\beta^*}\right)^n J(n), \quad (41)$$

where

$$J(n) = \int_0^1 dx \frac{x^2}{\sqrt{1-x^2}} \int_0^1 dy y^{n+1} \sqrt{1-x^2 y^2}. \quad (42)$$

The quantities  $J(n)$  for  $n = 0, 1, 2, \dots$  can be easily evaluated to yield

$$\begin{aligned} J(0) &= \frac{1}{3} \left(\frac{\pi}{2} - \frac{2}{3}\right), \\ J(1) &= \frac{G}{4} - \frac{1}{24}, \\ J(2) &= \frac{2}{15}, \end{aligned}$$

where  $G = 0.915\,965\,594\dots$  is the Catalan constant. We therefore for  $\beta^* \neq 0$  have

$$\varepsilon = -\frac{\hbar c}{2a^3\beta^{*2}} \left[ \frac{1}{3} \left( \frac{\pi}{2} - \frac{2}{3} \right) - \left( \frac{G}{4} - \frac{1}{24} \right) \frac{\pi}{\beta^*} + \frac{2\pi^2}{45\beta^{*2}} + \dots \right]. \quad (43)$$

The result of the numerical calculations using formula (37), for the dimensionless Casimir energy

$$\varepsilon_{2D}^* = \varepsilon \left/ \frac{\hbar c}{16\pi a^3} \zeta(3), \right.$$

is presented in figure 1 (curve 2D). As in the two- and three-dimensional cases, the Casimir energy  $\varepsilon$  goes to zero for  $\beta \rightarrow \infty$ .

## 5. Conclusions

We have shown that the deformation of the field given by equation (2) leads to suppression of the Casimir energy for the considered simple-topology surfaces localizing the field. The mechanism of this suppression is the following. The Casimir effect is determined by the difference between the energy of quantum fluctuations in the free configuration space and quantum fluctuation energy in the configurational space with corresponding boundary conditions. Just this difference corresponds to the suppression of Casimir energy by the deformation. This fact leads to an additional repulsion of the domain boundaries confining the field.

## Acknowledgment

The author appreciates interesting discussions with V Tkachuk, Yu Krynytskyi, T Fityo and A Rovenchak.

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